Selecting 2^{m-p} Designs Using a Minimum Aberration Criterion When Some Two-Factor Interactions Are Important

Weiming KE and Boxin TANG

Department of Mathematical Sciences,
University of Memphis
Memphis, TN 38152

We consider the problem of selecting appropriate 2^{m-p} designs when some two-factor interactions are important. Current methods in the literature select designs that permit estimation of the postulated model consisting of the main effects and important two-factor interactions, under the assumption that all of the other effects are negligible. When the effects not in the postulated model are not negligible, they will bias the estimates of the effects in the model. To minimize the contamination of these nonnegligible effects on the model, we propose and study a minimum aberration criterion. We then discuss the application of this new aberration criterion to compromise plans. Finally, we examine how to search for the best designs according to the criterion and present some results for designs of 16 and 32 runs.

KEY WORDS: Compromise plan; Design resolution; Fractional factorial design; Isomorphic model; Regular factorial design; Requirement set; Robust parameter design.

1. INTRODUCTION

Regular two-level fractional factorials, commonly referred to as 2^{m-p} designs, have been proven useful for efficient data collection in industrial experiments and other areas of scientific investigation. As exploratory designs, they are directly useful in identifying important factors at the early stage of an investigation. They form a basis on which other more sophisticated designs, such as central composite designs, can be built to study the complex relationship between a response variable and several explanatory variables.

In this article we consider how to select 2^{m-p} designs when some two-factor interactions (2fi's) are presumably important. Such a design problem arises in many applications. For example, in robust parameter design, interactions between control and noise factors are deemed important and should be estimated along with main effects. One way of solving this design problem is to find designs that allow joint estimation of all main effects and these presumably important 2fi's under the assumption that all other effects are negligible. Much work has been done on finding a design allowing estimation of a set of specified effects, often referred to as a *requirement set* in the literature (see, e.g., Greenfield 1976; Franklin 1985; Wu and Chen 1992). The work by Addelman (1962) on compromise plans is perhaps the earliest in this regard, although Addelman considered only some special collections of interactions.

When some 2fi's are important, the postulated model consists of all main effects and these important 2fi's. If the effects not in the postulated model cannot be completely ignored, then they will bias the estimates of the effects in the model. To solve this problem, we propose a minimum aberration criterion that systematically minimizes this bias caused by the nonnegligible effects. The criterion is more general than the usual minimum aberration criterion (Fries and Hunter 1980) and includes the latter as a special case.

Section 2 introduces and studies this new criterion of aberration and also discusses its application to compromise plans. Section 3 examines how to search for designs that are best according to the criterion and presents some results for designs of 16 and 32 runs. Section 4 concludes the article with a discussion.

2. A GENERAL CRITERION OF MINIMUM ABERRATION

2.1 Minimum Aberration and Its Robust Properties

A 2^{m-p} design has m factors, each at two levels, and $n = 2^{m-p}$ runs and is determined by p independent defining words. The two levels are denoted by $+1$ and -1, so the design matrix of such a design is an $n \times m$ matrix of ± 1. The defining relation of a 2^{m-p} design is the complete set of defining words. Labels of factors are referred to as "letters". A defining word specifies a set of letters that has the property that the product of the corresponding columns is a column of all $+1$'s. Including $\mathbf{I}$, the column of all 1's, the defining relation of a 2^{m-p} design has 2^p defining words. Let A_i be the number of defining words of length i in the defining relation, where the length of a word is the number of letters in the word. The minimum aberration criterion, as introduced by Fries and Hunter (1980), selects designs by sequentially minimizing $A_3, \dots, A_m$. The minimum aberration criterion has some attractive robust properties. Here we review a result from Tang and Deng (1999); for other robust properties, we refer to work of Cheng, Steinberg, and Sun (1999).

To begin, suppose that we are interested only in estimating all main effects. Then the fitted model is the main effect model, given by

$$\mathbf{Y} = \beta_0 \mathbf{I} + \mathbf{X}_1 \boldsymbol{\beta}_1 + \boldsymbol{\epsilon}, \tag{1}$$

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where $\mathbf{Y}$ denotes the vector of n observations, β_0 is the grand mean, $\mathbf{X}_1$ is the original design matrix $\mathbf{D}$, $\boldsymbol{\beta}_1$ is the vector of all main effects, and $\boldsymbol{\epsilon}$ is the vector of uncorrelated random errors, assumed to have mean 0 and a constant variance. Because interactions may not be negligible, the true model can be written as

$$\mathbf{Y} = \beta_0\mathbf{I} + \mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\boldsymbol{\beta}_2 + \cdots + \mathbf{X}_m\boldsymbol{\beta}_m + \boldsymbol{\epsilon}, \qquad (2)$$

where $\boldsymbol{\beta}_j$ is the vector of $\binom{m}{j}$ interactions involving j factors and $\mathbf{X}_j$ is the corresponding matrix obtained by taking all products of j columns from $\mathbf{X}_1 = \mathbf{D}$. The least squares solution $\hat{\boldsymbol{\beta}}_1 = (\mathbf{X}_1^T\mathbf{X}_1)^{-1}\mathbf{X}_1^T\mathbf{Y} = n^{-1}\mathbf{X}_1^T\mathbf{Y}$ from the fitted model in (1) has expectation, taken under the true model in (2), of $E(\hat{\boldsymbol{\beta}}_1) = \boldsymbol{\beta}_1 + \mathbf{C}_2\boldsymbol{\beta}_2 + \cdots + \mathbf{C}_m\boldsymbol{\beta}_m$, where $\mathbf{C}_j = n^{-1}\mathbf{X}_1^T\mathbf{X}_j$ for $j \geq 2$. So the bias of $\hat{\boldsymbol{\beta}}_1$ for estimating $\boldsymbol{\beta}_1$ is $\text{bias}(\hat{\boldsymbol{\beta}}_1, \boldsymbol{\beta}_1) = \mathbf{C}_2\boldsymbol{\beta}_2 + \cdots + \mathbf{C}_m\boldsymbol{\beta}_m$. Note that $\mathbf{C}_j\boldsymbol{\beta}_j$ is the contribution of $\boldsymbol{\beta}_j$ to the bias. Because $\boldsymbol{\beta}_j$ is unknown, we have to work with $\mathbf{C}_j$. One size measure for a matrix $\mathbf{C} = (c_{ij})$ is given by $\|\mathbf{C}\|^2 \stackrel{\text{def}}{=} \text{trace}(\mathbf{C}^T\mathbf{C}) = \sum_{i,j} c_{ij}^2$. Under the hierarchical assumption that lower-order effects are more important than higher-order effects, to minimize the bias of $\hat{\boldsymbol{\beta}}_1$, we should sequentially minimize $\|\mathbf{C}_2\|^2, \ldots, \|\mathbf{C}_m\|^2$. For regular designs, the entries of $\mathbf{C}_j$ are 0 or 1, and thus $\|\mathbf{C}_j\|^2$ is simply the number of j-factor interactions aliased with the main effects.

From Tang and Deng (1999), we have

$$\|\mathbf{C}_j\|^2 = (j+1)A_{j+1} + (m-j+1)A_{j-1} \qquad (3)$$

for $2 \leq j \leq m-1$ and $\|\mathbf{C}_m\|^2 = A_{m-1}$. The relationship in (3) leads to the following result.

Lemma 1. The minimum aberration criterion is equivalent to a criterion that sequentially minimizes $\|\mathbf{C}_2\|^2, \ldots, \|\mathbf{C}_m\|^2$, where $\|\mathbf{C}_j\|^2$ is the number of j-factor interactions aliased with the main effects.

Tang and Deng (1999) in fact presented their result for both regular and nonregular designs. For details, see their article.

2.2 A General Criterion of Aberration: Minimum *N*-Aberration

The idea embodied in Lemma 1 is quite rich. Lemma 1 establishes a statistical justification for the minimum aberration criterion. More importantly, Lemma 1 suggests a way of generalizing the minimum aberration criterion to the problem of design selection when some 2fi's are presumably important. This can be done straightforwardly by following the same path taken when deriving Lemma 1.

When some 2fi's are important, they should also be included in the fitted model, which is now given by

$$\mathbf{Y} = \beta_0\mathbf{I} + \mathbf{W}_1\boldsymbol{\gamma}_1 + \boldsymbol{\epsilon}, \qquad (4)$$

where $\boldsymbol{\gamma}_1$ is the vector of parameters containing all main effects and the important 2fi's and $\mathbf{W}_1$ is the corresponding matrix. The true model in (2) is now written as

$$\mathbf{Y} = \beta_0\mathbf{I} + \mathbf{W}_1\boldsymbol{\gamma}_1 + \mathbf{W}_2\boldsymbol{\gamma}_2 + \mathbf{X}_3\boldsymbol{\beta}_3 + \cdots + \mathbf{X}_m\boldsymbol{\beta}_m + \boldsymbol{\epsilon}, \qquad (5)$$

where $\boldsymbol{\gamma}_2$ is the vector of remaining 2fi's and $\mathbf{W}_2$ is the corresponding matrix. As before, the least squares estimate $\hat{\gamma_1}$ from the fitted model in (4) has a bias given by

$$\text{bias}(\hat{\boldsymbol{\gamma}}_1, \boldsymbol{\gamma}_1) = \mathbf{P}_2\boldsymbol{\gamma}_2 + \mathbf{P}_3\boldsymbol{\beta}_3 + \cdots + \mathbf{P}_m\boldsymbol{\beta}_m,$$

where $\mathbf{P}_2 = n^{-1}\mathbf{W}_1^T\mathbf{W}_2$ and $\mathbf{P}_j = n^{-1}\mathbf{W}_1^T\mathbf{X}_j$ for $j \geq 3$.

When some 2fi's are important, (i.e., the postulated model as in (4) is fitted), it is fairly natural to select designs using a *minimum aberration criterion*, defined as one that sequentially minimizes $\|\mathbf{P}_2\|^2, \ldots, \|\mathbf{P}_m\|^2$. Now let $N_j = \|\mathbf{P}_j\|^2$. Then N_j is the number of j-factor interactions aliased with the effects in the postulated model in (4). For convenience, we call the criterion that sequentially minimizes $N_2, \ldots, N_m$ the *minimum N-aberration criterion.*

Lemma 1 says that when γ_1 contains only the main effects, meaning that no 2fi's are postulated to be important, the minimum N-aberration criterion reduces to the usual minimum aberration criterion.

Another interesting case is where we want to estimate all 2fi's besides all main effects. Implicitly we are assuming that designs of resolution at least V exist. Because $\boldsymbol{\gamma}_1$ consists of all main effects and all 2fi's, $\boldsymbol{\gamma}_2$ in (5) vanishes. Hence the minimum N-aberration sequentially minimizes $N_3, \ldots, N_m$. Simple algebra reveals that

$$N_j = (j+1)A_{j+1} + (m-j+1)A_{j-1} + \binom{j+2}{2}A_{j+2} + j(m-j)A_j + \binom{m-j+2}{2}A_{j-2}, \qquad (6)$$

where we define $A_j = 0$ for $j > m$. Because designs of resolution V are under consideration, we have $A_j = 0$ for $1 \leq j \leq 4$. So (6) implies that $N_3 = 10A_5$, $N_4 = 5A_5 + \binom{6}{2}A_6$, and so on. It is now immediatly seen that sequentially minimizing $N_3, \ldots, N_m$ is equivalent to sequentially minimizing $A_5, \ldots, A_m$. We record this result in the following theorem.

Theorem 1. When all main effects and all two-factor interactions are to be estimated, the minimum N-aberration criterion, defined to be the one sequentially minimizing $N_3, \ldots, N_m$, is equivalent to the usual minimum aberration criterion that sequentially minimizes $A_5, \ldots, A_m$.

Wu and Chen (1992) suggested using the usual minimum aberration criterion, in combination with a graphical method, to select designs when some 2fi's are important. Designs selected in this way may not have the robust property in minimizing the contamination of nonnegligible effects on the postulated model. However, from Lemma 1 and Theorem 1, it is not unreasonable to use the usual minimum aberration criterion when the number of important 2fi's is close to 0 or $\binom{m}{2}$. We return to this point in Section 3.3.

To gain further insight into the minimum N-aberration, we now examine the criterion in detail. A design allows estimation of all main effects and all 2fi's if and only if it has resolution at least V—namely, it has no defining words of length 3 or 4. In general, a design allows estimation of the model in (4) if and only if none of its length-3 words contains an important 2fi and none of its length-4 words contains two important 2fi's that have no letter in common. This can be explained as follows. If a length-3 word contains an important 2fi, then this important 2fi is aliased with a main effect. Similarly, if a length-4 word contains two important 2fi's that have no letter in common, then the two important 2fi's are mutually aliased. The minimum N-aberration criterion selects designs, from among all designs satisfying the above condition for estimation, by sequentially minimizing $N_2, \ldots, N_m$. A general expression for N_j similar to those in (3) and (6) can be derived. Because this general result is fairly complicated, in this paper we present the result only for N_2 and N_3.

Let $A_4^{(2)} = \sum_{i=1}^{k} a_i$, where k denotes the number of important 2fi's, and a_i is the number of length-4 words containing the ith important 2fi. Then we have that

$$N_2 = 3A_3 + A_4^{(2)}. \tag{7}$$

Now define $A_5^{(2)}$ in the same way as $A_4^{(2)}$, but replacing length-4 words by length-5 words in the foregoing. Further, let $A_3^{(1)} = \sum_{i=1}^{k} b_i$, where b_i is the number of length-3 words containing just one letter in the ith important 2fi. Then it can easily be shown that

$$N_3 = 4A_4 + A_5^{(2)} + A_3^{(1)}. \tag{8}$$

Note that $A_4^{(2)}$ does not represent the number of length-4 words containing an important 2fi. This is because if a length-4 word contains more than one important 2fi, then it is counted more than once in calculating $A_4^{(2)}$. In fact, $A_4^{(2)}$ corresponds to the total number of times a length-4 word contains an important 2fi. A similar interpretation holds for $A_5^{(2)}$ and $A_3^{(1)}$. A simple example helps here. Consider a 2^{6-2} design given by $I = 125 = 2346 = 13456$. Suppose that the important 2fi's are 13, 23, and 24. Clearly, this design allows joint estimation of six main effects, (1, 2, 3, 4, 5, 6) and three important 2fi's, (13, 23, and 24). The design has only one length-4 word, (2346), but this word contains two important 2fi's, (23 and 24). Therefore, $A_4^{(2)} = 2$. For each of the three important 2fi's, (13, 23, and 24), the length-3 word, (125), contains exactly one of the two letters. Thus $A_3^{(1)} = 3$. The length-5 word, (13456), contains only one important 2fi (13), and thus $A_5^{(2)} = 1$.

The requirement set comprises all of the main effects and all of the important 2fi's. For the effects in the requirement set to be estimable, the important 2fi's in the requirement set cannot be aliased with each other and with main effects. This condition for estimation dictates that the design cannot have certain defining words of length 3 and 4, as discussed earlier. (We now look at the 2fi's not in the requirement set. In general, they cause a bias on the estimation of the effects in the requirement set. The measure of this bias, as given by N_2, is the number of the 2fi's outside the requirement set that are aliased with the effects in the requirement set. The first term, $3A_3$, in the expression for N_2 given in (7) represents the number of those 2fi's outside the requirement set that are aliased with the main effects, whereas the second term, $A_4^{(2)}$, represents the number of those 2fi's outside the requirement set that are aliased with the important 2fi's. If a design has $N_2 = 0$, then $A_3 = A_4^{(2)} = 0$, in which case the design is of resolution IV and the important 2fi's are clear. A 2fi is clear if and only if it is not aliased with any main effect and any other 2fi (Wu and Hamada 2000).

The minimum N-aberration criterion goes on to minimize N_3 once N_2 is minimized. Note that N_3 is the total number of 3fi's that are aliased with the effects in the requirement set. From (8), we see that N_3 is a sum of two components, with the first component, $4A_4$, representing the number of 3fi's that are aliased with the main effects and the second component, $A_5^{(2)} + A_3^{(1)}$, representing the number of 3fi's that are aliased with the important 2fi's.

2.3 Weak N-Aberration and Its Application to Compromise Plans

In practice, we are often quite confident that interactions involving three or more factors are negligible. In this case we are satisfied with only minimizing N_2 in finding the best designs. This gives a weak version of the minimum N-aberration criterion. We now apply the criterion of weak N-aberration to compromise plans. Three classes of compromise plans were discussed by Addelman (1962); the fourth class of compromise plans was discussed by Sun (1993). To describe these plans, suppose that the factors are divided into two groups, G_1 and G_2. We use $G_i \times G_i$ to denote the set of 2fi's among the factors in G_i, for $i = 1, 2$. Similarly, we use $G_1 \times G_2$ to denote the set of 2fi's between the factors in G_1 and those in G_2. Let A_{ij} be the number of words of length $i + j$ having i factors from G_1 and j factors from G_2 in the defining relation.

Table 1 presents the resulting N_2 for four classes of compromise plans. To illustrate, let us look at compromise plans of class 1. In this case the 2fi's in $G_1 \times G_1$ are supposed to be important, so the fitted model in (4) contains all main effects and all 2fi's in $G_1 \times G_1$. For this model to be estimable, it is necessary and sufficient that $A_{30} = A_{21} = A_{40} = 0$. Given that $A_{30} = A_{21} = A_{40} = 0$, we can verify that $N_2 = 3A_{12} + 3A_{03} + 3A_{31} + A_{22}$ in a fairly straightforward manner. The criterion of weak N-aberration is then to select a design that minimizes $N_2 = 3A_{12} + 3A_{03} + 3A_{31} + A_{22}$ from among all designs satisfying $A_{30} = A_{21} = A_{40} = 0$.

We now look at an example. Suppose that we want to estimate the 2fi between factor 1 and factor 2, denoted by 12, in addition to the main effects of six factors $(1, 2, \ldots, 6)$ using 16 runs. This is the case of compromise plans of class 1, where

Table 1. Weak Minimum N-Aberration for Four Classes of Compromise Plans

Class	Important 2fi's	Condition for estimation	N_2
1	$G_1 \times G_1$	$A_{30} = A_{21} = A_{40} = 0$	$3A_{12} + 3A_{03} + 3A_{31} + A_{22}$
2	$G_1 \times G_1$, $G_2 \times G_2$	$A_3 = A_{22} = A_{40} = A_{04} = 0$	$3A_{31} + 3A_{13}$
3	$G_1 \times G_1$, $G_1 \times G_2$	$A_{30} = A_{21} = A_{12} = A_{31} = A_{40} = A_{22} = 0$	$3A_{03} + 3A_{13}$
4	$G_1 \times G_2$	$A_{21} = A_{12} = A_{22} = 0$	$3A_{30} + 3A_{03} + 3A_{31} + 3A_{13}$

G_1 contains factors 1 and 2 and G_2 contains factors 3, 4, 5, and 6. Consider three designs, D_1, D_2, and D_3, where D_1 is given by $I = 125 = 1346 = 23456$, D_2 is given by $I = 1235 = 1246 = 3456$, and D_3 is given by $I = 1235 = 2346 = 1456$. For D_1, we have $A_{21} = 1$. So D_1 does not allow joint estimation of the effects 1, 2, 3, 4, 5, 6, and 12. Both D_2 and D_3 are of resolution IV, and thus $A_{30} = A_{21} = 0$ for both of them. It is also obvious that $A_{40} = 0$. So the postulated model is estimable under both designs. Because G_1 has only two factors, we have that $A_{31} = 0$. It is also true that $A_{12} = A_{03} = 0$, because both designs have resolution IV. We thus have $N_2 = A_{22}$, which equals 2 for D_2 and 1 for D_3. So D_3 is the best among the three designs under the minimum N-aberration criterion.

3. SEARCHING FOR MINIMUM N-ABERRATION DESIGNS

3.1 Nonisomorphic Models

The minimum N-aberration criterion, as defined in Section 2.2, depends on the fitted model. Let m be the number of factors. Then the total number of 2fi's is $K = \binom{m}{2}$. Any subset of these K 2fi's can be included in the fitted model, implying that the total number of models of the form in (4) that can be fit is 2^K, a very large number even for moderate m. For example, when $m = 8$, we have $2^K = 2^{28} = 268{,}435{,}456$. Noting that some of these models have essentially the same structure, we give a definition of isomorphic models. Two models are said to be isomorphic if one model can be obtained from the other by relabeling the factors. The graph theory is useful for finding the number of nonisomorphic models. If we associate a factor with a vertex and a 2fi with an edge, then a model of the form (4) defines a graph. Thus for given m, the total number of nonisomorphic models corresponds to the number of nonisomorphic graphs based on m vertices. For $m = 8$, this number is 12,346 (Wilson 1976, p. 162). We conclude that a catalog of minimum N-aberration designs for all nonisomorphic models would still be too large to be practically useful. Instead, we consider all nonisomorphic models containing up to four 2fi's. Our search method, discussed in the next section, can be easily generalized to deal with models with more than four 2fi's. In this article, we do not pursue this direction further, because the problem will inevitably become more complicated, both theoretically and computationally.

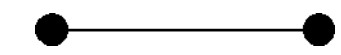

Figure 1. Graph for Model With One Two-Factor Interaction.

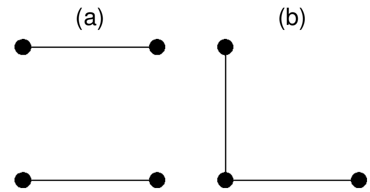

Figure 2. Graphs for Models With Two Two-Factor Interactions.

Let k be the number of important 2fi's. For $k = 1$, there is only one nonisomorphic model, as represented by Figure 1. For $k = 2$, 3, and 4, the number of nonisomorphic models is 2, 5, and 11, and the graphs for these nonisomorphic models are given in Figures 2, 3, and 4. Note that the graphs given here do not describe the models completely, because we have deliberately omitted the vertices that have no edges connected to them. Having those vertices in the graphs would require a separate set

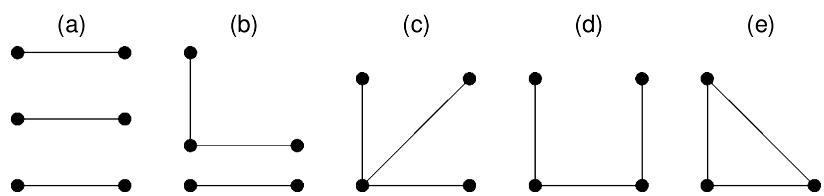

Figure 3. Graphs for Models With Three Two-Factor Interactions.

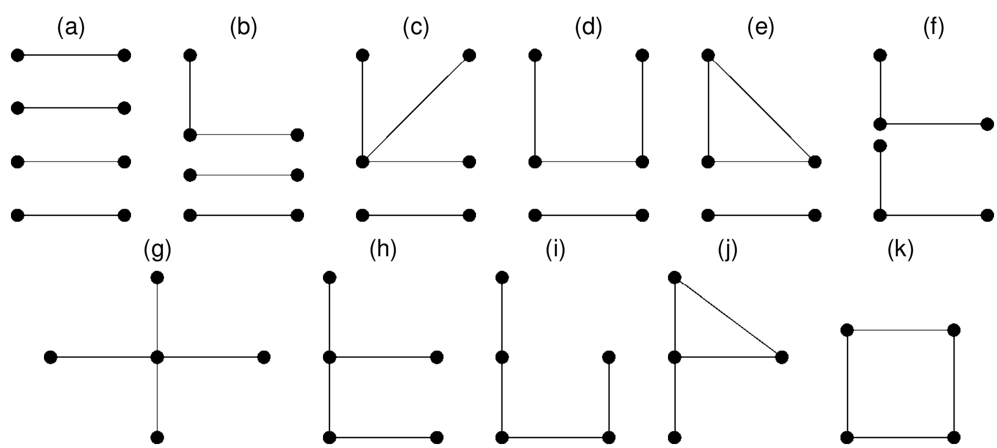

Figure 4. Graphs for Models With Four Two-Factor Interactions.

of graphs for each m. The treatment presented here allows us to present only one set of graphs and still retain the essential structures of the nonisomorphic models.

3.2 Search Method

Chen, Sun, and Wu (1993) gave a complete catalog of all nonisomorphic designs of 16 and 32 runs and all resolution IV nonisomorphic designs of 64 runs. This catalog can be used to generate all possible designs of 16 and 32 runs. Two designs are defined to be isomorphic if one design can be obtained from the other by permuting the columns, switching the signs, or a combination of these operations. For a given design in the catalog, different factor assignments may lead to different designs for the model under consideration. Although the total number of factor assignments is $m!$, with m being the number of factors, we need only consider those assignments that can possibly lead to different designs, which can be done with a simple analysis of the structure of the model under consideration. For example, consider the graph in Figure 2(b), representing a model containing two 2fi's that have one common factor. The number of ways of assigning this common factor is $\binom{m}{1}$; and the number of ways of assigning the other two interacting factors, is $\binom{m-1}{2}$. So the total number of assignments is $\binom{m}{1}\binom{m-1}{2}$, a much smaller number than $m!$. For the model in Figure 3(a), the number of factor assignments that we need to consider is $\binom{m}{2}\binom{m-2}{2}\binom{m-4}{2}/3!$.

Despite the foregoing simplification, the number of assignments can still become large for large m, making it difficult to conduct a complete search. So far, we have found all minimum N-aberration designs of 16 runs for $1 \le k \le 4$ and almost all minimum N-aberration designs of 32 runs for $1 \le k \le 3$, where k is the number of important 2fi's. In our search effort, we have used (N_2, N_3, N_4) instead of the entire vector $(N_2, \ldots, N_m)$, to further reduce the computing burden.

3.3 Minimum N-Aberration Designs of 16 and 32 Runs

Tables 2, 3, 4, and 5 present minimum N-aberration designs of 16 runs for models with one, two, three, and four important 2fi's, respectively. In Tables 3, 4, and 5, the entries under "model" indicate which model is under consideration; for example, the entry 2(a) denotes the model represented by Figure 2(a). The entries under "parent design" give the designs from which the minimum N-aberration designs are found; the design labels from Chen et al. (1993) are used here. These parent designs can be reconstructed based on the information in

Table 2. Minimum N-Aberration Designs of 16 Runs for the Model Containing One 2fi, as in Figure 1

m	Parent design	2fi	(N_2, N_3, N_4)
5	5-1.1	(1, 2)	(0, 1, 5)
6	6-2.1	(1, 4)	(1, 12, 2)
7	7-3.1	(1, 2)	(2, 28, 4)
8	8-4.1	(1, 2)	(3, 56, 8)
9	9-5.1	(2, 4)	(15, 60, 72)
10	10-6.1	(2, 8)	(27, 80, 148)
11	11-7.1	(1, 14)	(39, 117, 256)
12	12-8.1	(1, 6)	(53, 172, 408)
13	13-9.1	(2, 12)	(71, 242, 620)
14	14-10.1	(1, 14)	(90, 336, 924)

Table 3. Minimum N-Aberration Designs of 16 Runs for the Models Containing Two 2fi's, as in Figure 2

m	Model	Parent design	2fi's	(N_2, N_3, N_4)
5	2(a)	5-1.1	(1, 2) (4, 8)	(0, 2, 5)
	2(b)	5-1.1	(1, 2) (1, 4)	(0, 2, 5)
6	2(a)	6-2.1	(1, 4) (2, 8)	(2, 12, 4)
	2(b)	6-2.1	(1, 4) (2, 4)	(2, 12, 4)
7	2(a)	7-3.1	(1, 2) (4, 8)	(4, 28, 8)
	2(b)	7-3.1	(1, 2) (1, 4)	(4, 28, 8)
8	2(a)	8-4.1	(1, 2) (4, 8)	(6, 56, 16)
	2(b)	8-4.1	(1, 2) (1, 4)	(6, 56, 16)
9	2(a)	9-5.1	(2, 4) (3, 8)	(18, 64, 80)
	2(b)	9-5.1	(2, 4) (3, 4)	(18, 64, 80)
10	2(a)	10-6.1	(2, 8) (3, 14)	(30, 88, 160)
	2(b)	10-6.1	(2, 8) (3, 8)	(30, 88, 160)
11	2(a)	11-7.1	(1, 14) (2, 5)	(43, 129, 272)
	2(b)	11-7.1	(1, 13) (2, 13)	(43, 129, 272)
12	2(a)	12-8.1	(1, 10) (2, 5)	(58, 188, 432)
	2(b)	12-8.1	(1, 6) (1, 10)	(58, 188, 432)
13	2(a)	13-9.1	(2, 13) (4, 10)	(76, 264, 660)
	2(b)	13-9.1	(2, 12) (3, 12)	(76, 264, 660)

Table 4. Minimum N-Aberration Designs of 16 Runs for the Models Containing Three 2fi's, as in Figure 3

m	Model	Parent design	2fi's	(N_2, N_3, N_4)
5	3(a)	–	–	–
	3(b)	5-1.1	(1, 2) (4, 8) (4, 15)	(0, 3, 5)
	3(c)	5-1.1	(1, 2) (1, 4) (1, 8)	(0, 3, 5)
	3(d)	5-1.1	(1, 4) (4, 8) (8, 2)	(0, 3, 5)
	3(e)	5-1.1	(1, 2) (1, 4) (2, 4)	(0, 3, 5)
6	3(a)	6-2.1	(1, 4) (2, 8) (7, 11)	(3, 12, 6)
	3(b)	6-2.2	(2, 4) (3, 8) (3, 13)	(3, 10, 11)
	3(c)	6-2.2	(2, 4) (2, 8) (2, 13)	(3, 10, 11)
	3(d)	6-2.2	(2, 8) (8, 3) (3, 4)	(3, 10, 11)
	3(e)	6-2.1	(1, 4) (1, 8) (4, 8)	(3, 12, 6)
7	3(a)	7-3.1	(1, 4) (2, 8) (7, 11)	(6, 28, 12)
	3(b)	7-3.1	(1, 2) (4, 8) (7, 8)	(6, 28, 12)
	3(c)	7-3.1	(1, 2) (1, 4) (1, 7)	(6, 28, 12)
	3(d)	7-3.1	(1, 4) (4, 8) (8, 2)	(6, 28, 12)
	3(e)	7-3.1	(1, 2) (1, 4) (2, 4)	(6, 28, 12)
8	3(a)	8-4.1	(1, 4) (2, 8) (7, 11)	(9, 56, 24)
	3(b)	8-4.1	(1, 2) (4, 8) (7, 8)	(9, 56, 24)
	3(c)	8-4.1	(1, 2) (1, 4) (1, 7)	(9, 56, 24)
	3(d)	8-4.1	(1, 4) (4, 8) (8, 2)	(9, 56, 24)
	3(e)	8-4.1	(1, 2) (1, 4) (2, 4)	(9, 56, 24)
9	3(a)	9-5.1	(2, 4) (3, 8) (5, 9)	(21, 68, 88)
	3(b)	9-5.1	(2, 4) (3, 8) (5, 8)	(21, 68, 88)
	3(c)	9-5.1	(2, 4) (3, 4) (4, 8)	(21, 68, 88)
	3(d)	9-5.1	(2, 4) (4, 8) (8, 3)	(21, 68, 88)
	3(e)	9-5.1	(2, 4) (2, 8) (4, 8)	(21, 68, 88)
10	3(a)	10-6.1	(2, 9) (3, 14) (4, 8)	(33, 96, 172)
	3(b)	10-6.1	(4, 8) (2, 9) (3, 9)	(33, 96, 172)
	3(c)	10-6.1	(2, 8) (3, 8) (4, 8)	(33, 96, 172)
	3(d)	10-6.1	(2, 14) (14, 3) (3, 8)	(33, 96, 172)
	3(e)	10-6.1	(2, 5) (2, 8) (5, 8)	(34, 96, 168)
11	3(a)	11-7.1	(1, 14) (2, 9) (3, 4)	(47, 141, 288)
	3(b)	11-7.1	(3, 4) (1, 13) (2, 13)	(47, 141, 288)
	3(c)	11-7.1	(1, 14) (2, 14) (5, 14)	(47, 141, 288)
	3(d)	11-7.1	(1, 10) (10, 5) (5, 2)	(47, 141, 288)
	3(e)	11-7.1	(1, 6) (1, 10) (6, 10)	(48, 140, 284)
12	3(a)	12-8.2	(1, 12) (4, 11) (6, 8)	(63, 203, 457)
	3(b)	12-8.2	(4, 11) (1, 12) (2, 12)	(63, 203, 457)
	3(c)	12-8.2	(1, 12) (2, 12) (3, 12)	(63, 203, 457)
	3(d)	12-8.2	(4, 10) (10, 5) (5, 8)	(63, 203, 457)
	3(e)	12-8.1	(1, 6) (1, 10) (6, 10)	(63, 204, 456)

Note: A row with – indicates the situation where the specified model does not exist for the given number m of factors.

Table 5. Minimum N-Aberration Designs of 16 Runs for the Models Containing Four 2fi's, as in Figure 4

m	Model	Parent design	2fi's	(N_2, N_3, N_4)
5	4(a)	–	–	–
	4(b)	–	–	–
	4(c)	–	–	–
	4(d)	–	–	–
	4(e)	5-1.1	(1, 2) (4, 8) (4, 15) (8, 15)	(0, 4, 5)
	4(f)	–	–	–
	4(g)	5-1.2	(1, 8) (2, 8) (4, 8) (7, 8)	(0, 4, 4)
	4(h)	5-1.1	(1, 2) (2, 4) (2, 8) (8, 15)	(0, 4, 5)
	4(i)	5-1.1	(1, 4) (4, 8) (8, 15) (15, 2)	(0, 4, 5)
	4(j)	5-1.1	(1, 2) (2, 4) (2, 8) (4, 8)	(0, 4, 5)
	4(k)	5-1.1	(1, 2) (1, 8) (2, 4) (4, 8)	(0, 4, 5)
6	4(a)	–	–	–
	4(b)	–	–	–
	4(c)	6-2.2	(2, 4) (1, 8) (3, 8) (8, 13)	(5, 10, 10)
	4(d)	6-2.2	(2, 13) (13, 3) (3, 8) (1, 4)	(4, 11, 11)
	4(e)	6-2.2	(2, 4) (3, 8) (3, 13) (8, 13)	(4, 11, 11)
	4(f)	6-2.2	(1, 4) (2, 4) (3, 8) (3, 13)	(4, 11, 11)
	4(g)	6-2.1	(1, 8) (2, 8) (4, 8) (7, 8)	(4, 12, 8)
	4(h)	6-2.2	(2, 4) (2, 8) (2, 13) (3, 13)	(3, 12, 12)
	4(i)	6-2.2	(4, 2) (2, 13) (13, 3) (3, 8)	(3, 12, 12)
	4(j)	6-2.2	(2, 4) (3, 4) (3, 8) (4, 8)	(4, 11, 11)
	4(k)	6-2.2	(2, 4) (2, 8) (3, 4) (3, 8)	(3, 12, 12)
7	4(a)	–	–	–
	4(b)	7-3.1	(1, 2) (1, 4) (7, 8) (11, 13)	(8, 28, 16)
	4(c)	7-3.1	(1, 4) (2, 8) (7, 8) (8, 11)	(8, 28, 16)
	4(d)	7-3.2	(2, 14) (14, 1) (1, 8) (3, 4)	(8, 25, 26)
	4(e)	7-3.1	(1, 8) (2, 4) (2, 7) (4, 7)	(8, 28, 16)
	4(f)	7-3.1	(1, 2) (1, 4) (7, 8) (7, 11)	(8, 28, 16)
	4(g)	7-3.1	(1, 2) (1, 4) (1, 7) (1, 8)	(8, 28, 16)
	4(h)	7-3.2	(2, 8) (3, 8) (8, 1) (1, 14)	(8, 24, 26)
	4(i)	7-3.2	(2, 8) (8, 1) (1, 14) (14, 3)	(8, 24, 26)
	4(j)	7-3.1	(1, 2) (2, 4) (2, 8) (4, 8)	(8, 28, 16)
	4(k)	7-3.2	(1, 8) (1, 14) (2, 8) (2, 14)	(8, 24, 26)
8	4(a)	8-4.1	(1, 2) (4, 8) (7, 13) (11, 14)	(12, 56, 32)
	4(b)	8-4.1	(1, 2) (1, 4) (7, 8) (11, 13)	(12, 56, 32)
	4(c)	8-4.1	(1, 4) (2, 8) (7, 8) (8, 11)	(12, 56, 32)
	4(d)	8-4.1	(2, 8) (8, 11) (11, 7) (1, 4)	(12, 56, 32)
	4(e)	8-4.1	(1, 8) (2, 4) (2, 7) (4, 7)	(12, 56, 32)
	4(f)	8-4.1	(1, 2) (1, 4) (7, 8) (7, 11)	(12, 56, 32)
	4(g)	8-4.1	(1, 2) (1, 4) (1, 7) (1, 8)	(12, 56, 32)
	4(h)	8-4.1	(1, 2) (2, 4) (2, 7) (7, 8)	(12, 56, 32)
	4(i)	8-4.1	(1, 4) (4, 7) (7, 8) (8, 2)	(12, 56, 32)
	4(j)	8-4.1	(1, 2) (2, 4) (2, 8) (4, 8)	(12, 56, 32)
	4(k)	8-4.1	(1, 2) (1, 8) (2, 4) (4, 8)	(12, 56, 32)
9	4(a)	9-5.1	(2, 4) (3, 8) (5, 15) (9, 14)	(24, 72, 96)
	4(b)	9-5.1	(2, 4) (2, 8) (3, 14) (5, 9)	(24, 72, 96)
	4(c)	9-5.1	(2, 5) (3, 8) (4, 8) (8, 14)	(24, 72, 96)
	4(d)	9-5.1	(3, 9) (9, 5) (5, 8) (2, 4)	(24, 72, 96)
	4(e)	9-5.1	(3, 4) (2, 8) (2, 14) (8, 14)	(24, 72, 96)
	4(f)	9-5.1	(2, 4) (3, 4) (5, 8) (5, 9)	(24, 72, 96)
	4(g)	9-5.1	(2, 8) (3, 8) (4, 8) (5, 8)	(24, 72, 96)
	4(h)	9-5.1	(2, 4) (3, 4) (4, 8) (5, 8)	(24, 72, 96)
	4(i)	9-5.1	(2, 5) (5, 3) (3, 8) (8, 4)	(24, 72, 96)
	4(j)	9-5.1	(2, 4) (3, 4) (3, 8) (4, 8)	(24, 72, 96)
	4(k)	9-5.1	(2, 4) (2, 8) (3, 4) (3, 8)	(24, 72, 96)
10	4(a)	10-6.1	(2, 8) (3, 14) (4, 15) (5, 9)	(36, 104, 184)
	4(b)	10-6.2	(1, 13) (2, 13) (3, 4) (6, 8)	(37, 102, 184)
	4(c)	10-6.1	(1, 6) (2, 8) (3, 8) (4, 8)	(37, 104, 180)
	4(d)	10-6.2	(2, 13) (13, 3) (3, 4) (1, 10)	(37, 102, 184)
	4(e)	10-6.1	(2, 8) (3, 4) (3, 15) (4, 15)	(37, 104, 180)
	4(f)	10-6.1	(2, 8) (3, 8) (4, 9) (5, 9)	(36, 104, 184)
	4(g)	10-6.1	(2, 8) (3, 8) (4, 8) (5, 8)	(36, 104, 184)
	4(h)	10-6.1	(2, 8) (3, 8) (8, 4) (4, 9)	(36, 104, 184)
	4(i)	10-6.2	(2, 13) (13, 1) (1, 10) (10, 4)	(37, 102, 184)
	4(j)	10-6.1	(2, 8) (3, 4) (3, 8) (4, 8)	(37, 104, 180)
	4(k)	10-6.1	(2, 8) (2, 14) (3, 8) (3, 14)	(36, 104, 184)
11	4(a)	11-7.2	(1, 10) (2, 12) (4, 9) (7, 8)	(51, 152, 305)
	4(b)	11-7.1	(3, 8) (4, 8) (1, 14) (2, 5)	(51, 153, 304)
	4(c)	11-7.2	(2, 12) (1, 10) (5, 10) (7, 10)	(51, 152, 305)
	4(d)	11-7.1	(1, 13) (13, 2) (2, 9) (3, 4)	(51, 153, 304)
	4(e)	11-7.1	(1, 14) (2, 5) (2, 9) (5, 9)	(51, 153, 304)
	4(f)	11-7.2	(1, 12) (2, 12) (3, 8) (7, 8)	(51, 152, 305)
	4(g)	11-7.3	(4, 8) (5, 8) (6, 8) (7, 8)	(51, 152, 304)
	4(h)	11-7.2	(1, 12) (2, 12) (12, 3) (3, 8)	(51, 152, 305)
	4(i)	11-7.2	(2, 12) (12, 1) (1, 10) (10, 5)	(51, 152, 305)
	4(j)	11-7.1	(1, 14) (2, 5) (2, 14) (5, 14)	(51, 153, 304)
	4(k)	11-7.3	(4, 8) (4, 10) (5, 8) (5, 10)	(51, 152, 304)

Note: A row with – indicates the situation where the specified model does not exist for the given number m of factors.

Table 6, which provides the design columns for each design. Column j in Table 6 denotes the jth column in the 16-run saturated design with its columns arranged in Yates order, which can be written as

$$(a_1, a_2, a_1a_2, a_3, a_1a_3, a_2a_3, a_1a_2a_3, a_4, a_1a_4, a_2a_4, a_1a_2a_4, a_3a_4, a_1a_3a_4, a_2a_3a_4, a_1a_2a_3a_4), \quad (9)$$

where a_1, a_2, a_3, and a_4 are four independent columns. The entries under "2fi's" show how to assign the factors involved in the important 2fi's. The last column in Tables 2, 3, 4, and 5 gives (N_2, N_3, N_4).

Tables 7, 8, and 9 present minimum N-aberration designs of 32 runs for models with one, two, and three important 2fi's. Table 10 gives the parent designs of 32 runs.

It is worth mentioning that most of the parent designs in Tables 2–5 and 7–9 are minimum aberration designs. This finding is consistent with the observation in Section 2.2 that minimum aberration provides a good approximation to minimum N-aberration when the number of important 2fi's is small. It should be kept in mind, however, that when a parent design has minimum aberration, it alone does not specify a minimum N-aberration design; the information on factor assignments is needed to complete specification of the minimum N-aberration design. In fact, minimum N-aberration designs for several different models often have the same minimum aberration design as their parent. For a given model, when the parent design has minimum aberration, other factor assignments besides that given in the minimum N-aberration design do not necessarily have minimum N-aberration. An example helps here. For $m = 6$

Table 6. The Parent Designs in Tables 2, 3, 4, and 5

m	Parent design	Additional columns
5	5-1.1	15
	5-1.2	7
6	6-2.1	7 11
	6-2.2	3 13
7	7-3.1	7 11 13
	7-3.2	3 5 14
8	8-4.1	7 11 13 14
9	9-5.1	3 5 9 14 15
10	10-6.1	3 5 6 9 14 15
	10-6.2	3 5 6 9 10 13
11	11-7.1	3 5 6 9 10 13 14
	11-7.2	3 5 6 7 9 10 12
	11-7.3	3 5 6 7 9 10 11
12	12-8.1	3 5 6 9 10 13 14 15
	12-8.2	3 5 6 7 9 10 11 12
13	13-9.1	3 5 6 7 9 10 11 12 13
14	14-10.1	3 5 6 7 9 10 11 12 13 14

Note: Here each design contains the four independent columns 1, 2, 4, and 8 besides those additional columns.

Table 7. Minimum N-Aberration Designs of 32 Runs for the Model Containing One 2fi, as in Figure 1

m	Parent design	2fi	(N_2, N_3, N_4)
6	6-1.1	(1, 2)	(0, 0, 1)
7	7-2.1	(1, 8)	(0, 5, 11)
8	8-3.1	(1, 16)	(0, 14, 22)
9	9-4.1	(1, 29)	(0, 28, 43)
10	10-5.1	(1, 4)	(1, 44, 86)
11	11-6.1	(2, 13)	(2, 100, 23)
12	12-7.1	(1, 16)	(3, 152, 32)
13	13-8.1	(1, 8)	(4, 220, 45)
14	14-9.1	(1, 4)	(5, 308, 62)
15	15-10.1	(1, 2)	(6, 420, 84)
16	16-11.1	(1, 2)	(7, 560, 112)
17	17-12.1	(2, 4)	(31, 568, 784)
18	18-13.1	(2, 8)	(55, 608, 1480)
19	19-14.1	(2, 16)	(79, 680, 2240)
20	20-15.1	(2, 29)	(103, 784, 3105)
21	21-16.2	(2, 23)	(127, 926, 4112)
22	22-17.1	(1, 26)	(151, 1104, 5304)
23	23-18.1	(2, 28)	(175, 1323, 6720)
24	24-19.1	(1, 6)	(203, 1576, 8376)
25	25-20.1	(2, 12)	(239, 1844, 10328)
26	26-21.1	(2, 20)	(275, 2160, 12636)
27	27-22.1	(1, 30)	(311, 2525, 15360)
28	28-23.1	(1, 14)	(349, 2940, 18536)
29	29-24.1	(2, 28)	(391, 3402, 22204)
30	30-25.1	(1, 30)	(434, 3920, 26460)

Table 8. Minimum N-Aberration Designs of 32 Runs for the Models Containing Two 2fi's, as in Figure 2

m	Model	Parent design	2fi's	(N_2, N_3, N_4)
6	2(a)	6-1.1	(1, 2) (4, 8)	(0, 0, 2)
	2(b)	6-1.1	(1, 2) (1, 4)	(0, 0, 2)
7	2(a)	7-2.1	(1, 8) (2, 16)	(0, 6, 12)
	2(b)	7-2.1	(1, 8) (2, 8)	(0, 6, 12)
8	2(a)	8-3.1	(1, 16) (2, 29)	(0, 16, 24)
	2(b)	8-3.1	(1, 16) (2, 16)	(0, 16, 24)
9	2(a)	9-4.2	(1, 16) (2, 30)	(0, 34, 43)
	2(b)	9-4.1	(1, 29) (2, 29)	(0, 32, 46)
10	2(a)	10-5.1	(1, 4) (2, 8)	(2, 48, 92)
	2(b)	10-5.1	(1, 4) (2, 4)	(2, 48, 92)
11	2(a)	11-6.1	(2, 21) (4, 11)	(4, 100, 46)
	2(b)	11-6.1	(2, 25) (4, 25)	(4, 100, 46)
12	2(a)	12-7.1	(1, 16) (2, 21)	(6, 152, 64)
	2(b)	12-7.1	(1, 16) (2, 16)	(6, 152, 64)
13	2(a)	13-8.1	(1, 8) (2, 13)	(8, 220, 90)
	2(b)	13-8.1	(1, 8) (2, 8)	(8, 220, 90)
14	2(a)	14-9.1	(1, 4) (2, 8)	(10, 308, 124)
	2(b)	14-9.1	(1, 4) (2, 4)	(10, 308, 124)
15	2(a)	15-10.1	(1, 2) (4, 8)	(12, 420, 168)
	2(b)	15-10.1	(1, 2) (1, 4)	(12, 420, 168)
16	2(a)	16-11.1	(1, 2) (4, 8)	(14, 560, 224)
	2(b)	16-11.1	(1, 2) (1, 4)	(14, 560, 224)
17	2(a)	17-12.1	(2, 4) (3, 8)	(38, 576, 896)
	2(b)	17-12.1	(2, 4) (3, 4)	(38, 576, 896)
18	2(a)	18-13.1	(2, 8) (3, 14)	(62, 624, 1600)
	2(b)	18-13.1	(2, 8) (3, 8)	(62, 624, 1600)
19	2(a)	19-14.1	(2, 16) (3, 22)	(86, 704, 2376)
	2(b)	19-14.1	(2, 16) (3, 16)	(86, 704, 2376)
20	2(a)	20-15.1	(2, 29) (3, 14)	(110, 817, 3265)
	2(b)	20-15.1	(2, 28) (3, 28)	(110, 817, 3265)
21	2(a)	21-16.2	(2, 27) (3, 22)	(134, 968, 4304)
	2(b)	21-16.2	(2, 27) (5, 27)	(134, 968, 4304)
22	2(a)	22-17.1	(1, 29) (2, 25)	(158, 1156, 5536)
	2(b)	22-17.1	(1, 26) (6, 26)	(158, 1156, 5536)
23	2(a)	23-18.1	(1, 6) (2, 28)	(185, 1379, 6980)
	2(b)	23-18.1	(2, 28) (3, 28)	(185, 1379, 6980)
24	2(a)	24-19.1	(1, 10) (2, 5)	(214, 1640, 8688)
	2(b)	24-19.1	(1, 6) (1, 10)	(214, 1640, 8688)
25	2(a)	25-20.1	(2, 12) (3, 20)	(250, 1920, 10704)
	2(b)	25-20.1	(2, 12) (3, 12)	(250, 1920, 10704)
26	2(a)	26-21.1	(2, 20) (3, 26)	(286, 2248, 13088)
	2(b)	26-21.1	(2, 20) (3, 20)	(286, 2248, 13088)
27	2(a)	27-22.1	(1, 30) (2, 13)	(323, 2625, 15888)
	2(b)	27-22.1	(1, 25) (6, 25)	(323, 2625, 15888)
28	2(a)	28-23.1	(1, 22) (2, 13)	(362, 3052, 19152)
	2(b)	28-23.1	(1, 14) (1, 22)	(362, 3052, 19152)
29	2(a)	29-24.1	(2, 29) (4, 26)	(404, 3528, 22932)
	2(b)	29-24.1	(2, 28) (3, 28)	(404, 3528, 22932)

Table 9. Minimum N-Aberration Designs of 32 Runs for the Models Containing Three 2fi's, as in Figure 3

m	Model	Parent design	2fi's	(N_2, N_3, N_4)
6	3(a)	6-1.1	(1, 2) (4, 8) (16, 31)	(0, 0, 3)
	3(b)	6-1.1	(1, 2) (4, 8) (4, 16)	(0, 0, 3)
	3(c)	6-1.1	(1, 2) (1, 4) (1, 8)	(0, 0, 3)
	3(d)	6-1.1	(1, 4) (4, 8) (8, 2)	(0, 0, 3)
	3(e)	6-1.1	(1, 2) (1, 4) (2, 4)	(0, 0, 3)
7	3(a)	7-2.1	(1, 8) (2, 16) (4, 27)	(0, 7, 13)
	3(b)	7-2.1	(1, 8) (2, 16) (4, 16)	(0, 7, 13)
	3(c)	7-2.1	(1, 8) (2, 8) (4, 8)	(0, 7, 13)
	3(d)	7-2.1	(1, 16) (16, 2) (2, 8)	(0, 7, 13)
	3(e)	7-2.1	(1, 8) (1, 16) (8, 16)	(0, 8, 12)
8	3(a)	8-3.1	(1, 4) (2, 16) (7, 29)	(1, 17, 26)
	3(b)	8-3.1	(1, 16) (2, 29) (4, 29)	(0, 18, 26)
	3(c)	8-3.1	(1, 16) (2, 16) (4, 16)	(0, 18, 26)
	3(d)	8-3.1	(1, 29) (29, 2) (2, 16)	(0, 18, 26)
	3(e)	8-3.1	(1, 16) (1, 29) (16, 29)	(0, 20, 24)
9	3(a)	9-4.1	(1, 4) (2, 8) (7, 29)	(2, 32, 51)
	3(b)	9-4.2	(1, 16) (2, 30) (4, 30)	(0, 37, 47)
	3(c)	9-4.1	(1, 29) (2, 29) (4, 29)	(0, 36, 49)
	3(d)	9-4.2	(1, 30) (30, 2) (2, 16)	(0, 37, 47)
	3(e)	9-4.2	(1, 16) (1, 30) (16, 30)	(0, 41, 43)
10	3(a)	10-5.1	(1, 4) (2, 8) (7, 11)	(3, 52, 98)
	3(b)	10-5.1	(1, 4) (2, 8) (7, 8)	(3, 52, 98)
	3(c)	10-5.1	(1, 4) (2, 4) (4, 8)	(3, 52, 98)
	3(d)	10-5.1	(1, 4) (4, 8) (8, 2)	(3, 52, 98)
	3(e)	10-5.1	(1, 4) (1, 8) (4, 8)	(3, 52, 98)
11	3(a)	11-6.1	(2, 25) (4, 19) (7, 8)	(6, 100, 69)
	3(b)	11-6.1	(7, 8) (2, 25) (4, 25)	(6, 100, 69)
	3(c)	11-6.4	(1, 8) (1, 14) (1, 16)	(6, 91, 117)
	3(d)	11-6.1	(2, 21) (21, 11) (11, 4)	(6, 100, 69)
	3(e)	11-6.1	(2, 4) (2, 25) (4, 25)	(7, 100, 65)
12	3(a)	12-7.1	(1, 19) (2, 21) (4, 16)	(9, 152, 96)
	3(b)	12-7.1	(4, 11) (1, 16) (2, 16)	(9, 152, 97)
	3(c)	12-7.3	(1, 16) (1, 22) (1, 26)	(9, 139, 169)
	3(d)	12-7.1	(1, 21) (21, 2) (2, 16)	(9, 152, 96)
	3(e)	12-7.1	(1, 14) (1, 16) (14, 16)	(9, 152, 97)
13	3(a)	13-8.1	(1, 11) (2, 13) (4, 8)	(12, 220, 135)
	3(b)	13-8.1	(4, 8) (1, 11) (2, 11)	(12, 220, 135)
	3(c)	13-8.2	(1, 14) (1, 22) (1, 26)	(12, 203, 239)
	3(d)	13-8.1	(1, 13) (13, 2) (2, 8)	(12, 220, 135)
	3(e)	13-8.1	(1, 8) (1, 16) (8, 16)	(12, 220, 135)

and the model with one important 2fi, Table 2 shows that the minimum N-aberration design, which has $N_2 = 1$, can be obtained by making $(1, 4)$ the important 2fi in the minimum aberration design 6-2.1. However, if the important 2fi is assigned to $(1, 2)$, then the resulting design has $N_2 = 2$.

3.4 An Illustrative Example

Suppose that in an industrial experiment, the experimenter studies seven factors: *temperature*, *moisture*, *pressure*, *thickness*, *time*, *size*, and *speed*. She would like to use a two-level design of 16 runs. In addition to the main effects of these factors, she also wants to estimate the two 2fi's between *temperature* and *moisture* and between *moisture* and *time*. The graph for this model is 2(b) as in Figure 2. The minimum N-aberration design for this model can be found in Table 3. Now let us look at the row for $m = 7$ and model 2(b) in Table 3. We see that the parent design is design 7-3.1, which, according to Table 6, collects columns 1, 2, 4, 8, 7, 11, and 13 from the saturated design of 16 runs in (9). To complete the specification of the minimum N-aberration design, we need to appropriately assign the seven factors to the seven columns. The 2fi's column in Table 3 says that we should assign *moisture* to column 1, and assign *temperature* and *time* to columns 2 and 4. Other factors can be

Table 9. (continued.)

14	3(a)	14-9.1	(1, 4) (2, 8) (7, 11)	(15, 308, 186)
	3(b)	14-9.1	(1, 4) (2, 8) (7, 8)	(15, 308, 186)
	3(c)	14-9.2	(1, 22) (1, 26) (1, 28)	(15, 286, 331)
	3(d)	14-9.1	(1, 4) (4, 8) (8, 2)	(15, 308, 186)
	3(e)	14-9.1	(1, 4) (1, 8) (4, 8)	(15, 308, 186)
15	3(a)	15-10.1	(1, 4) (2, 8) (7, 11)	(18, 420, 252)
	3(b)	15-10.1	(1, 2) (4, 8) (7, 8)	(18, 420, 252)
	3(c)	15-10.1	(1, 2) (1, 4) (1, 7)	(18, 420, 252)
	3(d)	15-10.1	(1, 4) (4, 8) (8, 2)	(18, 420, 252)
	3(e)	15-10.1	(1, 2) (1, 4) (2, 4)	(18, 420, 252)
16	3(a)	16-11.1	(1, 4) (2, 8) (7, 11)	(21, 560, 336)
	3(b)	16-11.1	(1, 2) (4, 8) (7, 8)	(21, 560, 336)
	3(c)	16-11.1	(1, 2) (1, 4) (1, 7)	(21, 560, 336)
	3(d)	16-11.1	(1, 4) (4, 8) (8, 2)	(21, 560, 336)
	3(e)	16-11.1	(1, 2) (1, 4) (2, 4)	(21, 560, 336)
17	3(a)	17-12.1	(2, 8) (3, 14) (4, 15)	(45, 584, 1008)
	3(b)	17-12.1	(4, 16) (2, 8) (3, 8)	(45, 584, 1008)
	3(c)	17-12.1	(2, 4) (3, 4) (4, 8)	(45, 584, 1008)
	3(d)	17-12.1	(2, 4) (4, 8) (8, 3)	(45, 584, 1008)
	3(e)	17-12.1	(2, 4) (2, 8) (4, 8)	(45, 584, 1008)
18	3(a)	18-13.1	(1, 6) (2, 8) (3, 14)	(70, 640, 1712)
	3(b)	18-13.1	(1, 6) (2, 8) (3, 8)	(70, 640, 1712)
	3(c)	18-13.1	(2, 8) (3, 8) (4, 8)	(69, 640, 1720)
	3(d)	18-13.1	(2, 14) (14, 3) (3, 8)	(69, 640, 1720)
	3(e)	18-13.1	(2, 8) (2, 16) (8, 16)	(69, 640, 1720)
19	3(a)	19-14.1	(1, 6) (2, 16) (3, 22)	(94, 728, 2504)
	3(b)	19-14.1	(1, 6) (2, 16) (3, 16)	(94, 728, 2504)
	3(c)	19-14.1	(2, 16) (3, 16) (4, 16)	(93, 728, 2512)
	3(d)	19-14.1	(2, 22) (22, 3) (3, 16)	(93, 728, 2512)
	3(e)	19-14.1	(2, 15) (2, 16) (15, 16)	(93, 729, 2512)
20	3(a)	20-15.1	(1, 6) (2, 29) (3, 14)	(118, 849, 3417)
	3(b)	20-15.1	(1, 6) (2, 28) (3, 28)	(118, 849, 3417)
	3(c)	20-15.1	(2, 29) (3, 29) (4, 29)	(117, 850, 3425)
	3(d)	20-15.1	(2, 28) (28, 3) (3, 14)	(117, 850, 3425)
	3(e)	20-15.1	(2, 5) (2, 27) (5, 27)	(118, 850, 3416)

assigned arbitrarily to the remaining columns. This design has $N_2 = 4$, meaning that four 2fi's not in the model are aliased with the effects in the model. Because design 7-3.1 is of resolution IV, aliasing between main effects and 2fi's does not occur. Therefore, these four 2fi's are aliased with the important 2fi's.

Now suppose that the experimenter would like to include two more 2fi's, the ones between *moisture* and *pressure* and between *size* and *speed*, in her requirement set. Then her new model is 4(c) in Figure 4. The minimum N-aberration design for this model can be found at the row for $m = 7$ and model 4(c) in Table 5. The parent design is the same design 7-3.1. The minimum N-aberration design can be obtained by assigning *moisture* to column 8; assigning *temperature*, *time*, and *pressure* to columns 2, 7, and 11; and assigning *size* and *speed* to columns 1 and 4. Other factors can be assigned arbitrarily to the remaining columns.

4. DISCUSSION

In robust parameter design, both control factors and noise factors are studied, and the objective is to choose the settings of control factors that are insensitive to the noise factors. The information on the 2fi's between control and noise factors is particularly useful for achieving this objective. By treating the 2fi's between control and noise factors as important, our method based on the minimum N-aberration criterion is directly applicable to such problems. Previously, Wu and Zhu (2003) proposed a similar aberration criterion for robust parameter design, based on their effect ordering principle. On the other hand, if only some of the 2fi's between control and noise factors are considered important, which is often the case in practical applications, our method still can be applied.

Besides analyzing location effects, it is important to analyze dispersion effects in robust parameter design. Suitable designs should, therefore, support analysis of both location and dispersion effects. Building on the early work by Rosenbaum

Table 10. The Parent Designs in Tables 7, 8, and 9

m	*Parent design*	*Additional columns*
6	6-1.1	31
7	7-2.1	7 27
8	8-3.1	7 11 29
9	9-4.1	7 11 19 29
	9-4.2	7 11 13 30
10	10-5.1	7 11 19 29 30
11	11-6.1	7 11 13 19 21 25
	11-6.4	3 5 14 22 26 29
12	12-7.1	7 11 13 14 19 21 25
	12-7.3	3 5 9 14 22 26 29
13	13-8.1	7 11 13 14 19 21 22 25
	13-8.2	3 5 9 14 17 22 26 28
14	14-9.1	7 11 13 14 19 21 22 25 26
	14-9.2	3 5 9 14 15 17 22 26 28
15	15-10.1	7 11 13 14 19 21 22 25 26 28
16	16-11.1	7 11 13 14 19 21 22 25 26 28 31
17	17-12.1	3 5 9 14 15 17 22 23 26 27 28 29
18	18-13.1	3 5 6 9 14 15 17 22 23 26 27 28 29
19	19-14.1	3 5 6 9 10 14 15 17 22 23 26 27 28 29
20	20-15.1	3 5 6 9 10 14 15 17 18 22 23 26 27 28 29
21	21-16.2	3 5 6 9 10 13 14 15 17 18 22 23 26 27 28 29
22	22-17.1	3 5 6 9 10 13 14 15 17 18 21 22 23 25 26 29 30
23	23-18.1	3 5 6 9 10 13 14 15 17 18 21 22 23 25 26 27 28 29
24	24-19.1	3 5 6 9 10 13 14 15 17 18 21 22 23 25 26 27 28 29 30
25	25-20.1	3 5 6 7 9 10 11 12 13 17 18 19 20 21 26 27 28 29 30 31
26	26-21.1	3 5 6 7 9 10 11 12 13 14 17 18 19 20 21 26 27 28 29 30 31
27	27-22.1	3 5 6 7 9 10 11 12 13 14 17 18 19 20 21 22 25 26 27 28 29 30
28	28-23.1	3 5 6 7 9 10 11 12 13 14 17 18 19 20 21 22 25 26 27 28 29 30 31
29	29-24.1	3 5 6 7 9 10 11 12 13 14 15 17 18 19 20 21 22 23 24 25 26 27 28 29
30	30-25.1	3 5 6 7 9 10 11 12 13 14 15 17 18 19 20 21 22 23 24 25 26 27 28 29 30

Note: Here each design contains the five independent columns 1, 2, 4, 8, and 16 besides those additional columns.

(1996), Hedayat and Stufken (1999) provided a catalog of designs that support analysis of both location and dispersion effects. To combine the strength of their designs with that offered by our method, one can apply the N-aberration criterion to their designs.

In this article, we have taken the view that the effects in the requirement set, consisting of the main effects and important 2fi's, are equally important. (For the effects not in the requirement set, we assume that the hierarchical ordering principle applies.) If one feels that the main effects need more protection than the important 2fi's, then the bias of the main effects should be minimized first. This suggests a variation of the N-aberration criterion, which sequentially minimizes $N_{21}, N_{22}, N_{31}, N_{32}, \ldots,$ where N_{j1} is the number of j-factor interactions aliased with the main effects and N_{j2} is the number of j-factor interactions aliased with the important 2fi's. Clearly, $N_j = N_{j1} + N_{j2}$, where N_j, as defined in Section 2.2, is the number of j-factor interactions aliased with the effects in the requirement set. From (7) in Section 2.2, we see that $N_{21} = 3A_3$ and $N_{22} = A_4^{(2)}$. So this modified N-aberration criterion minimizes A_3 first, just as the usual minimum aberration does. If the minimum aberration design is the only one (up to isomorphism) that minimizes A_3, then the best designs under the modified criterion must come from the minimum aberration design. There are many interesting research problems associated with the modified criterion, and these will be explored in future work.

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