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Optimal partition of sampling effort between observations of fish density and migration speed for a riverine hydroacoustic duration-in-beam method

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Abstract

The duration-in-beam model has long been used in fisheries acoustics. For instance, the Pacific Salmon Commission uses this model for the hydroacoustic data collected at Mission, BC, Canada to estimate daily salmon passages in the Fraser River for in-season salmon management. The fish flux is calculated as the product of fish density times the migration speed. These two components are measured separately with the same acoustical sampling system. As the system can only acquire either density or speed data at any given time, an optimal sampling design is needed to partition the sampling effort for acoustical measurements of fish density and migration speed. In this paper, we present a method to estimate the optimal allocation of sampling effort between the density and speed measurements. The optimum was achieved by minimizing the variance of the estimated fish flux. The theoretical optimization method was applied to the hydroacoustic data collected in the 1998 fishing season to obtain an estimated optimal sampling design. This optimal design was adopted for the hydroacoustic sampling program at Mission, BC in the 1999 salmon migration season. The statistical results from the 1999 data were examined and compared with that from the 1998 data acquired according to a non-optimized sampling plan. It was found that the optimized sampling plan reduced the relative variance by 20%. Therefore, the methodology of this optimal design for sampling effort is a useful addition to the duration-in-beam model in fisheries acoustics.

Keywords: Sampling optimization; Fish flux; Duration-in-beam; Acoustical estimates

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1. Introduction

A duration-in-beam hydroacoustic sampling method has been used for estimating the escapement of sockeye salmon (*Oncorhynchus* nerka) to the Fraser River by the Pacific Salmon Commission (PSC) for many years in relation to in-season management of salmon fisheries (Woodey, 1987). Fig. 1 shows the location of the town of Mission (the acoustic site where upstream migration salmon are estimated) and a portion of the Fraser River watershed, a major salmon production system on the west coast of North America. The escapement estimates are derived from echo data collected with a downward-looking, single-beam sonar and are based on the duration-in-beam model proposed by Thorne (1988), and refined by Banneheka

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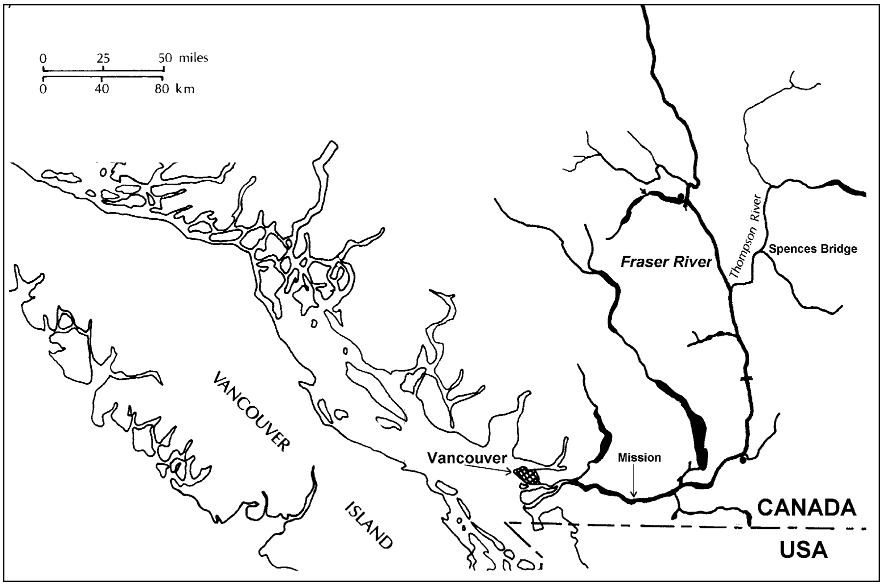

Fig. 1. Map showing the lower Fraser River watershed and approximate location of PSCs hydroacoustic site at Mission, BC. Shaded areas are sockeye salmon rearing lakes.

et al. (1995), hereafter referred to as Banneheka. This model estimates daily passages of salmon past the acoustic site by representing the fish flux as the product of the estimates of fish density times migration speed. The density estimates come from measurements made from a boat that conducts shore-to-shore transects across the river. However, the migration speed estimates are obtained from stationary soundings at randomly chosen locations within the area covered by the transects. To minimize measurement bias, it is desirable to collect both types of echo data with the same transducer and echo sounder mounted on the same vessel. Therefore, the total sampling effort must be divided between these two types of sampling, mobile sampling for density and stationary sampling for migration speed.

The variance of the estimate for daily fish passage depends on (among other things) the number of transects per day, the average number of fish seen per transect, the number of fish seen in the migration speed measurements and the migration speed. Because the total sampling time available is limited, there is a tradeoff between sampling effort spent monitoring density and effort spent monitoring speed. In 1998, we were asked by the PSC to investigate an optimal partition of sampling effort between observations of fish density and migration speed in order to minimize the estimated variance for the fish passage, which was left unfinished from Banneheka's work. However, Banneheka realized in their discussion that "A scheme for optimal allocation of sampling effort between the two types of soundings could substantially reduce the coefficient of variation". We follow the discussion by Banneheka and explore a method to minimize the variance of the estimated daily fish passage by optimally partitioning this sampling effort. It should be noted that errors in measurement of fish behavior by the acoustical system can also contribute to estimation variance and bias. However, this type of variance and bias will be addressed in a separate study. The focus of the present paper is on minimizing estimation variance inherent to the duration-in-beam model with an optimal allocation for the sampling effort between the stationary and mobile observations for the improvement of the duration-in-beam model. The methodology in this paper can be readily applied to any form of fishery acoustics as a useful addition to the duration-in-beam method in planing and designing the sampling effort. We shall use our data as an example.

2. Materials and methods

2.1. Experimental procedures

Details of the experimental procedures and of the model formulation are described by Banneheka. We shall only mention the factors central to our objective of reducing the variance of the estimates of daily fish passage.

Since 1977, the Pacific Salmon Commission (PSC) has been using a single-beam echo sounder system (Biosonics Model 105) to collect fish echoes from a survey vessel at Mission during the annual fishing season. The 3 dB (or half-power) full-beam-width of the single-beam transducer is approximately 32°, and the unit has been calibrated yearly since 1995 in the calibration tank at BioSonics, Incorporated (Seattle, Washington). The transducer has shown a stable beam-width around 32° with a standard deviation of less than 2°. A 12 bit digitizer and a differential GPS receiver were integrated with the echo sounder system in 1996, which improved accuracy of duration-in-beam measurements, and allowed the positioning of individual targets on a fixed Cartesian frame. Fig. 2 shows a typical cumulative cross-river fish distribution over an 18 h period at the site.

The echo sounder system transmits approximately 15 sound pulses per second into the water. Each pulse is 0.8 ms long with a central frequency of 50 kHz. The digital form of the return signals are stored in a computer for further processing. The analogue form of the signals are recorded as echograms (range from the transducer versus time) on chart paper by a Biosonics Model 111 thermal chart recorder with a paper speed of 0.478 mm s^{-1}. All of the echoes recorded as a target passes through the acoustic beam form a characteristic pattern on the echogram called a “trace”. Each trace is counted as an individual fish. Estimating fish density from traces is a well known method in the field of fisheries acoustics. For stationary soundings, the width of the trace on the chart recording is related to the duration-in-beam of the fish, which is related to the fish’s migration speed and its distance from the transducer. In practice, Banneheka uses the inverse of the measured trace width in his formulae and we shall do the same.

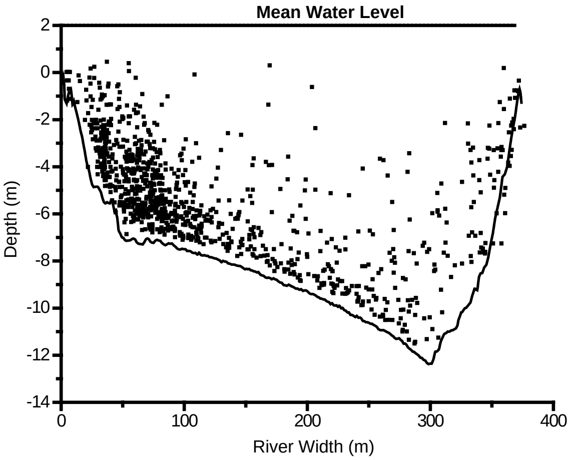

Fig. 2. Cumulative cross-river target distribution based on 18 h of transect sampling of echo data on August 12, 1998. A total of 1200 targets were identified as migratory fish. Filled squares indicate mean locations of individual fish traces, and the thick line outlines the bottom profile of the river along a primary transect-bearing of 320° based on 216 transects. Note: some transect-bearings were deviated from the primary bearing making the corresponding detected targets appear below the plotted profile. The 2 m blind-zone below the river surface resulted from a finite deployment depth of the transducer and its near-field blanking distance. A 0.5 m blind-zone also exists near the river bottom due to a finite beam-width of the transducer.

Historically, the time spent per day recording the acoustic data has been about 21 h. The remaining 3 h are required for crew changes, refueling, etc. Within the 21 h sampling period, typically 18 h are devoted to density measurements from the moving boat and 3 h are devoted to duration-in-beam measurements when the boat is stationary. Since each transect takes approximately 5 min to complete, data from about 216 transects could be obtained per day. The stationary soundings are stratified into three zones, north side, south side and middle, depending on location in the river. Each day, three random locations are selected within each of these strata. Thus, nine stationary soundings are conducted every day, each sounding taking approximately 20 min. We shall use the same mathematical notation as Banneheka to avoid confusion. All of the measurements (both stationary and mobile) are stratified into one of three depth strata, 0–5, 5–10, and 10–15 m. The model assumes that fish are distributed randomly with a uniform density within each depth stratum. It also assumes that fish speed has a random and common distribution within each stratum. All of the transect and stationary soundings from a single day are used to calculate the passage rate of migrating fish. Banneheka shows that, under these conditions, the number of fish passing upstream per second can be estimated by

$$\hat{R} = a\sum_{i=1}^{3}\bar{N}_i\bar{M}_i \quad (\mathrm{s}^{-1}), \tag{1}$$

where $\bar{N}_i$ is the average of the number of traces per transect in the ith depth stratum, $\bar{M}_i$ (mm^{-1}) the average of the inverse of the trace widths detected in the ith depth stratum during stationary soundings, and a ($\mathrm{mm\,s}^{-1}$) is a constant related to the paper speed of the chart recorder. The term of $a\bar{N}_i\bar{M}_i$ is the estimated number of fish passing upstream per second in the ith depth stratum, denoted as $\hat{R}_i$. Note that Eq. (1) assumes that fish density and migration speed are independent (Banneheka et al., 1995). The estimated number of fish passing the acoustic site per day is $\hat{R}$ times 86 400, the number of seconds per day. We shall now examine the problem of how to minimize the variance of $\hat{R}$ by adjusting time spent obtaining transect data as compared to that spent obtaining stationary sounding data.

2.2. Optimization of sampling effort

The theoretical variance of the estimate given in Eq. (1) is shown by Banneheka to be

$$\begin{aligned}\mathrm{Var}(\hat{R}) &= \mathrm{Var}\left(\sum_{i=1}^{3}\hat{R}_i\right) = a^2\sum_{i=1}^{3}\mathrm{Var}(\bar{N}_i\bar{M}_i)\\ &= a^2\sum_{i=1}^{3}[\mathrm{Var}(\bar{N}_i)E^2(\bar{M}_i) + \mathrm{Var}(\bar{M}_i)E^2(\bar{N}_i)\\ &\quad + \mathrm{Var}(\bar{N}_i)\mathrm{Var}(\bar{M}_i)],\end{aligned} \tag{2}$$

where Var denotes the variance and E denotes the statistical expectation with E^2 being the square of the expectation. The term $\mathrm{Var}(\hat{R})$ has dimension of s^{-2}. The expression following the second equal-sign in (2) results from the assumption that fish flux in the ith depth stratum is uncorrelated with that in the jth stratum ($i \neq j$), which is reasonable based on the swimming behavior of migratory salmon. As the dominating direction of travel for the majority of fish is upstream (Xie et al., 2002), there is little exchange in fish flux in the depth direction, i.e., across different stratum layers on a horizontal sampling area of less than 400 m × 100 m. The assumption that fish density and migration speed are independent guarantees the rest of the derivations in (2).

The total sampling time per day in seconds, T, is composed of the sum of the time spent in the two types of soundings, therefore it is subject to the following constraint:

$$t_0 n + \tau = T, \tag{3}$$

where t_0 is the time (s) for a transect survey (in our case $t_0 = 300$ s), n the total number of transect surveys per day and τ the total time in seconds of stationary surveys per day. The number of fish observed in the ith stratum after τ seconds of stationary sampling is:

$$m_i = \tau p_i = (T - t_0 n)p_i, \tag{4}$$

where p_i is the number of fish traces in the ith stratum observed on the chart recording per second of sampling. Using standard notation for both the mean (μ) and variance (σ^2) of N_i and M_i, Eq. (2) can be expressed as:

$$\begin{aligned}\mathrm{Var}(\hat{R}) &= a^2\sum_{i=1}^{3}\left[\frac{\sigma_{N_i}^2\sigma_{M_i}^2}{nm_i} + \frac{\sigma_{N_i}^2}{n}\mu_{M_i}^2 + \frac{\sigma_{M_i}^2}{m_i}\mu_{N_i}^2\right]\\ &= a^2\sum_{i=1}^{3}\left[\frac{\sigma_{N_i}^2\sigma_{M_i}^2}{n(T - t_0 n)p_i} + \frac{\sigma_{N_i}^2}{n}\mu_{M_i}^2\right.\\ &\quad\left. + \frac{\sigma_{M_i}^2}{(T - t_0 n)p_i}\mu_{N_i}^2\right]\\ &= a^2\left[\frac{b_1}{n(T - t_0 n)} + \frac{b_2}{nT} + \frac{b_3}{T - t_0 n}\right]\\ &= \frac{a^2}{t_0}\left[\frac{b_1}{n(L - n)} + \frac{b_2}{nL} + \frac{b_3}{L - n}\right] \quad (\mathrm{s}^{-2}),\end{aligned} \tag{5}$$

where $L = T/t_0$ is the maximum number of transects, i.e., $0 < n < L$, and

$$b_1 = \sum_{i=1}^{3}\frac{\sigma_{N_i}^2\sigma_{M_i}^2}{p_i}, \qquad b_2 = \sum_{i=1}^{3}\sigma_{N_i}^2\mu_{M_i}^2 T,$$

$$b_3 = \sum_{i=1}^{3}\frac{\sigma_{M_i}^2\mu_{N_i}^2}{p_i}.$$

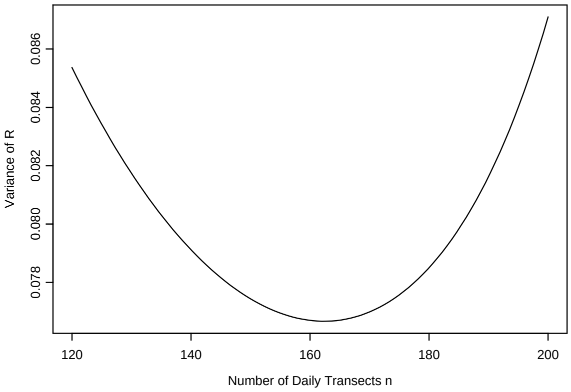

Fig. 3. Estimated variance of fish flux in (5) as a function of n based on the data taken from August 30, 1998.

The factors b_1, b_2 and b_3 all have a dimension of $\mathrm{mm}^{-2}\,\mathrm{s}$. Eqs. (5) and (3) demonstrate how $\mathrm{Var}(\hat{R})$ depends on the partitioning of sampling effort between transect and stationary surveys. Specifically, for a given T, it is the number of transect surveys, n, that determines the variance of $\hat{R}$. Therefore, the necessary condition for minimizing (5) is $\mathrm{d}(\mathrm{Var}(\hat{R}))/\mathrm{d}n = 0$. When this is done and the resulting equation is solved for n, we obtain:

$$n_{\mathrm{opt}} = L\frac{(b_1+b_2) - \sqrt{(b_1+b_2)^2-(b_1+b_2)(b_2-Lb_3)}}{b_2-Lb_3}. \tag{6}$$

Appendix A provides detailed derivations for obtaining (6) and also shows that the n given in (6) is the optimal n that minimizes $\mathrm{Var}(\hat{R})$ in the bound of $0 < n < L$. Fig. 3 graphically illustrates the trend for the $\mathrm{Var}(\hat{R})$ in (5) for given values of n, based on the data obtained on August 30, 1998. Eq. (6) is a function of the unknown parameters μ_{N_i}, μ_{M_i}, $\sigma^2_{N_i}$, $\sigma^2_{M_i}$ and p_i ($i = 1$–3). We can use the unbiased estimates $\bar{N}_i$, $\bar{M}_i$, $S^2_{N_i}$, $S^2_{M_i}$ and $\hat{p}_i$ to obtain an estimate of n_{opt} as:

$$\hat{n}_{\mathrm{opt}} = L\frac{(\hat{b}_1+\hat{b}_2) - \sqrt{(\hat{b}_1+\hat{b}_2)^2-(\hat{b}_1+\hat{b}_2)(\hat{b}_2-L\hat{b}_3)}}{\hat{b}_2-L\hat{b}_3}, \tag{7}$$

where

$$\hat{b}_1 = \sum_{i=1}^{3}\frac{S^2_{N_i}S^2_{M_i}}{\hat{p}_i}, \qquad \hat{b}_2 = \sum_{i=1}^{3} S^2_{N_i}\bar{M}_i^2 T,$$
$$\hat{b}_3 = \sum_{i=1}^{3}\frac{S^2_{M_i}\bar{N}_i^2}{\hat{p}_i}, \qquad S^2_{N_i} = \frac{1}{n-1}\sum_{k=1}^{n}(N_{i,k}-\bar{N}_i)^2,$$
$$S^2_{M_i} = \frac{1}{m_i-1}\sum_{k=1}^{m_i}(M_{i,k}-\bar{M}_i)^2, \qquad \hat{p}_i = \sum_{k=1}^{9}\frac{m_{i,k}}{\tau}. \tag{8}$$

In the above equations, k indexes individual observations of the corresponding variables.

3. Results

3.1. Analysis of the 1998 data

In 1998, data from 90 days are available that cover the period from June 24 (Julian day 175) to September 21 (Julian day 264). In this year, sampling effort for the transect sounding was chosen to be 18 h, which resulted in approximately 212 daily transect soundings, i.e., $n = 212$. The sampling effort left for the stationary soundings is then approximately 3 h. Within a day, the sample averages for the number of fish targets per transect N_i, and the average of inverse target width for stationary soundings M_i, are given by:

$$\bar{N}_i = \frac{1}{n}\sum_{k=1}^{n} N_{i,k}, \tag{9}$$

and

$$\bar{M}_i = \frac{1}{m_i}\sum_{k=1}^{m_i} M_{i,k} = \frac{1}{(T-t_0 n)p_i}\sum_{k=1}^{m_i} M_{i,k}, \tag{10}$$

where k indexes the subsamples from N_i and M_i. The sample variances, $S^2_{N_i}$ and $S^2_{M_i}$, are given in (8). The $\bar{N}_i$, $\bar{M}_i$, $S^2_{N_i}$ and $S^2_{M_i}$ provide unbiased estimates of μ_{N_i}, μ_{M_i}, $\sigma^2_{N_i}$ and $\sigma^2_{M_i}$.

During the 1998 salmon run, low fish density observed early in the run caused difficulties. On five separate occasions (June 25, July 1, 4, 11, and 15), only one fish was observed for some of the $\bar{M}_i$ values. Thus, no estimate of $S^2_{M_i}$ is possible for these days, nor is the $\bar{M}_i$ very reliable. To overcome these difficulties, the PSC employed the expedient of using the average from the previous 6 days to estimate these two parameters. All of the other observations, such as the $\bar{M}_i$ values for other strata and the $\bar{N}_i$, were obtained using data from only the pertinent day.

When (7) is applied to the 90 days of data from 1998 the resulting $\hat{n}_{\text{opt}}$-values can be determined (Fig. 4). For these data, $\hat{n}_{\text{opt}}$ ranges from 60 to 218. The mean $\hat{n}_{\text{opt}}$ is 160, which corresponds to a transect sampling effort of 13 h. However, the distribution of $\hat{n}_{\text{opt}}$ is skewed with the median value equal to 172 and the modal value equal to 180 (Fig. 5). Thus, the optimum transect sampling effort should fall between 13 and 15 h.

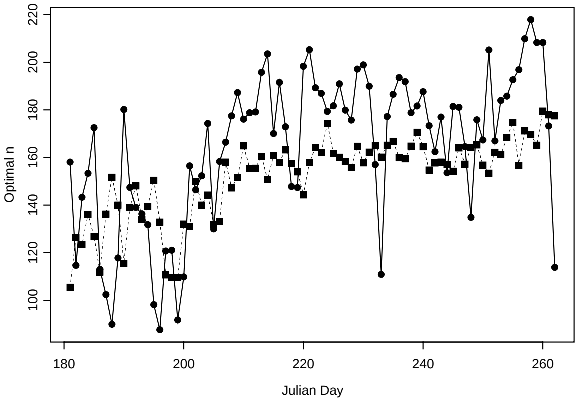

Fig. 4. The estimated optimal $\hat{n}_{\text{opt}}$ for each day is plotted for both 1998 and 1999 daily estimates. The filled circles connected by a solid line are for 1998 and the filled squares connected by a dashed line are for 1999.

3.2. Analysis of the 1999 data

Based on the analysis of the 1998 data, the PSC was suggested to adjust the 1999 sampling effort to 15 h for transect sampling and 6 h for stationary sounding. Data were collected from June 30 to September 19 for in-season management of the fishery. Post-season analysis examined the $\hat{n}_{\text{opt}}$-values (Fig. 4), which shows the same type of skewness as was observed for the 1998 data. The estimates have less variation for 1999 than for 1998, with a range from 105 to 180, mean of 150, and mode of 160 (Fig. 5).

In addition, we examined the unbiased estimate for $\mathrm{Var}(\hat{R})$ in Eq. (2), which is given by Banneheka as:

$$\widehat{\mathrm{Var}}(\hat{R}) = a^2 \sum_{i=1}^{3} \left(\frac{S_{N_i}^2}{n} \bar{M}_i^2 + \frac{S_{M_i}^2}{m_i} \bar{N}_i^2 - \frac{S_{N_i}^2 S_{M_i}^2}{n m_i} \right). \tag{11}$$

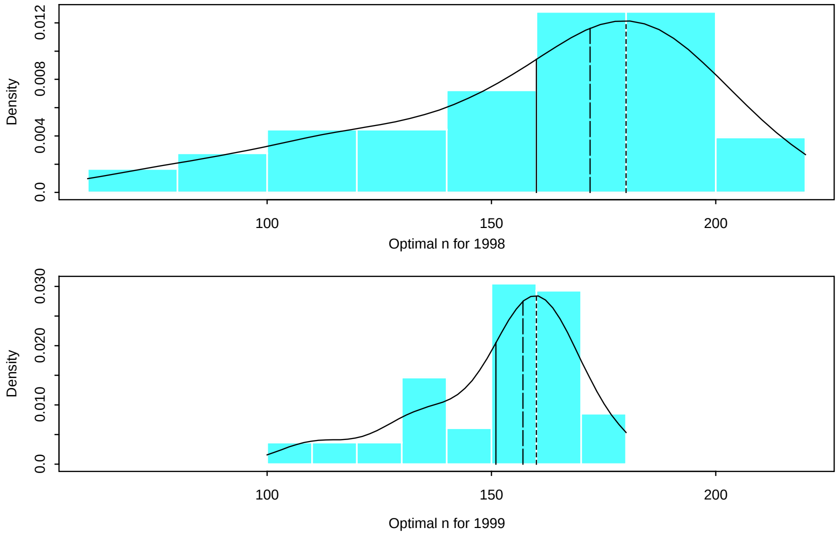

Fig. 5. Histograms of the distribution of the optimal value of $\hat{n}_{\text{opt}}$ for 1998 and 1999 daily estimates. The curved line is the smoothed density estimation. The vertical lines from left to right are the mean, median and mode, respectively.

It should be noted that the sign of the third term in Eq. (2) is opposite to that of the corresponding term in Eq. (11) as Eq. (11) is an unbiased estimate of the theoretical variance $\mathrm{Var}(\hat{R})$ given in Eq. (2) (or Eq. (5)). This can be easily proved by using the estimated variance of a product (Goodman, 1960; Kendall and Stuart, 1977).

When the $\widehat{\mathrm{Var}}(\hat{R})$ values are compared, the results are lower for 1999 (mean of $\widehat{\mathrm{Var}}(\hat{R}) = 3.02 \times 10^{-3}$) than for 1998 (mean of $\widehat{\mathrm{Var}}(\hat{R}) = 8.36 \times 10^{-3}$), which indicates that the mean variance for 1999 is only one-third of the variance of 1998. However, it was argued that the abundance of fish was lower in 1999 (daily average = 74 152) than in 1998 (daily average = 105 091). Therefore, we compared the relative error for these 2 years using the coefficient of variation (CV). The 1999 data had a mean CV of 0.072, while for 1998 the mean value was 0.086. Thus, there is a decrease of 20% in the mean of the relative error after the sampling effort was changed.

Table 1
Mean and standard deviation (STDEV) for the optimal n obtained from 1998 to 1999

	1998		1999	
	Early	Latter	Early	Latter
Mean	131	178	134	162
STDEV	35	23	15	7

The “Early” and “Latter” stages is divided by the Julian day 210.

Table 2
Daily average of the relative variances for 1998 and 1999 data

Stratum	1998		1999	
	CV_n	CV_m	CV_n	CV_m
1	2.0808	0.3149	1.4343	0.4118
2	1.2863	0.4187	1.1069	0.4799
3	2.1032	0.3975	1.2273	0.4936

CV_n is the coefficient of variation for fish density estimation and CV_m is the coefficient of variation for fish speed estimation.

4. Discussion

We have examined the potential for reducing the variance of the estimated daily fish passage by optimizing the sampling effort between transect and stationary soundings for the duration-in-beam model. We derived an expression for the optimal number of transects per day that would result in the minimum variance for the estimated daily fish passage. Using the data from 1998 as a starting point, we estimated that doubling the amount of time spent on the stationary soundings, where fish migration speed is estimated, would improve the precision of the estimates of fish passage. This change in sampling took place in 1999 and resulted in a decrease of 20% in the relative error for that year as compared to 1998. Therefore, it appears that the adjusted sampling effort did improve the estimates by reducing variability.

A close examination of the time series of $\hat{n}_{\mathrm{opt}}$-values (Fig. 4 and Table 1) reveals that there is a systematic pattern for both 1998 and 1999. The first portion of the migration season (up to Julian day 210) has lower values for $\hat{n}_{\mathrm{opt}}$ and higher variance than is observed for the remainder of the season. This may be the result of low abundance for the first portion of the run. If the data are divided into two portions using the boundary as day 210, a modified sampling approach may be appropriate (Table 1). This suggests a two-stage sampling effort scheme that would lead to 11 h of transect sampling for the first portion and 13 h for the latter portion. This modified scheme would triple the stationary sampling for the first portion and double it for the latter portion, when compared to the pre-1999 sampling design. The optimized sampling effort still favors the transect soundings, in terms of total time per day. This is consistent with the observation that fish density is more variable over a 24 h period than migration speed (Xie, personal communication). Table 2 also illustrates this fact based on the data from 1998 to 1999.

We hope that these modifications of the sampling will help to improve the precision of the sockeye salmon escapement estimates for the Fraser River. We fully appreciate that acoustical measurement errors can cause estimation bias and variance, and we have been working towards minimizing this type of error by adopting state-of-the-art sonar technologies for monitoring migratory salmon past Mission, BC (Mulligan and Chen, 1998; Xie, 2000). Improved management of this valuable natural resource may result from improved estimation methodology. In addition, the methodology for the optimal sampling effort discussed in this paper is a useful addition to the duration-in-beam model used in fishery acoustics.

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Appendix A. Derivation of Eq. (6)

It can be easily obtained from (5) with $\mathrm{d}(\mathrm{Var}(\hat{R}))/\mathrm{d}n = 0$ that

$$n_{1,2} = L\frac{(b_1+b_2) \pm\sqrt{(b_1+b_2)^2-(b_1+b_2)(b_2-Lb_3)}}{b_2-Lb_3}. \tag{A.1}$$

There are two solutions in Eq. (A.1), and both of them lead to positive second derivative of $\mathrm{Var}(\hat{R})$, i.e., $\mathrm{d}^2(\mathrm{Var}(\hat{R}))/\mathrm{d}n^2|_{n_{1,2}} > 0$. Therefore, n_1 and n_2 both minimize $\mathrm{Var}(\hat{R})$. However, it can be proved that only n_2 falls in the range of $0 < n < L$. To show this, we write $h = (b_2 - Lb_3)/(b_1 + b_2) = (1 - Lb_3/b_2)/(1 + b_1/b_2)$. Since all b_1, b_2 and b_3 are positive, we have $h < 1$. With notation h, Eq. (A.1) is

$$n_{1,2} = L\frac{1\pm\sqrt{1-h}}{h}. \tag{A.2}$$

Substitute $k = \sqrt{1-h}$, i.e. $h = 1 - k^2$. The two solutions in Eq. (A.2) are simplified as

$$\begin{aligned} n_{1,2} &= L\frac{1\pm k}{1-k^2} = L\frac{1\pm k}{(1+k)(1-k)} \\ &= \frac{L}{1-k} \quad \text{or} \quad \frac{L}{1+k}. \end{aligned} \tag{A.3}$$

Only $n_2 = L/(1 + k)$ is in the interval of $0 < n < L$, which corresponds to choosing the solution in Eq. (A.1) with the negative sign. This result is the n_{opt} given in Eq. (6).

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